

A TROPICAL KRULL-SCHMIDT THEOREM

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ABSTRACT. Continuing the study of semifield kernels, We develop some algebraic structure notions such as composition series and convexity degree, along with some notions holding a geometric interpretation, like reducibility and hyperdimension.

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1. OVERVIEW

Continuing the study of semifield kernels in tropical mathematics initiated in the doctoral dissertation of the first author and [21, 23], we turn to the basic question of a tropical Krull-Schmidt theory. Throughout, F denotes a ν -archimedean ν -semifield[†] (to be defined shortly), which from Proposition 3.30 onwards is assumed to be divisible, and $F(\Lambda)$ is the ν -semifield[†] of fractions of the polynomial semiring[†] $F[\Lambda]$ in the indeterminates $\Lambda = \{\lambda_1, \dots, \lambda_n\}$.

The most intuitive way to develop algebraic geometry over a semiring would be to consider the coordinate semiring of a variety, cf. [5, 19, 15]. But unfortunately homomorphisms of semirings are defined by congruences, not ideals, and the study of congruences is much more difficult than that of ideals. This led the authors in [23] to look for an alternative algebraic structure, namely that of semifield kernels.

As seen via [23, Chapter 6] and in particular Theorem 2.36, tropical varieties correspond to coordinate ν -semifields[†] (Definition 2.35), and thus, in view of Proposition 2.38, chains of kernels of $F(\Lambda)$, in particular, HP-kernels give us an algebraic tropical notion of dimension. Its determination is the subject of this paper, in which we focus on irreducible HP-kernels, which comprise the **hyperspace-spectrum**, cf. Definition 3.10. We get to the main results in §3.4, using standard Krull-Schmidt theory, proving catenarity in Theorem 3.50, and concluding that $F(\Lambda)$ has dimension n in Theorem 3.52 and Corollary 3.53.

Our tools include convexity degree, along with some notions having a geometric interpretation, such as reducibility and hyperdimension.

2. BACKGROUND

We recall the main ideas of [23], starting with a general review.

2.1. Semirings without zero.

Definition 2.1. A **semiring**[†] (semiring without zero) is a set $R := (R, +, \cdot, 1)$ equipped with binary operations $+$ and \cdot and distinguished element $\mathbf{1}_R$ such that:

- (i) $(R, +)$ is an Abelian semigroup;
- (ii) $(R, \cdot, \mathbf{1}_R)$ is a monoid with identity element $\mathbf{1}_R$;
- (iii) Multiplication distributes over addition.
- (iv) R contains elements r_0 and r_1 with $r_0 + r_1 = \mathbf{1}_R$.

A **domain**[†] is a commutative semiring[†] whose multiplicative monoid is cancellative.

Definition 2.2. A **semifield**[†] is a domain[†] in which every element is (multiplicatively) invertible.

(In other words, the multiplicative monoid is an Abelian group.) We need a fundamental correspondence between ordered monoids and semirings[†], [30, §4]:

Remark 2.3. Any semiring[†] can be viewed as a (multiplicative) semi-lattice ordered Abelian monoid, where we define

$$(2.1) \quad a \vee b := a + b.$$

Thus, we have a natural partial order given by $a \geq b$ whenever $a = b + c$ for some c . (This partial order is trivial for rings, but not for idempotent semirings!)

Conversely, any semi-lattice ordered Abelian monoid M becomes a semiring[†], where multiplication is the given monoid operation and addition is given by

$$(2.2) \quad a + b := a \vee b$$

(viewed in M).

2.2. Supertropical ν -semifields[†].

We bring in the “ghost” notation.

Definition 2.4. [23, Definition 3.2.1] A ν -**domain**[†] is a quadruple $(R, \mathcal{T}, \nu, \mathcal{G})$ where R is a semiring[†] and $\mathcal{T} \subset R$ is a cancellative multiplicative submonoid and $\mathcal{G} \triangleleft R$ is endowed with a partial order, together with an idempotent homomorphism $\nu : R \rightarrow \mathcal{G}$, with $\nu|_{\mathcal{T}}$ onto, satisfying the conditions:

$$\begin{aligned} a + b &= a \quad \text{whenever} \quad \nu(a) > \nu(b). \\ a + b &= \nu(a) \quad \text{whenever} \quad \nu(a) = \nu(b). \end{aligned}$$

\mathcal{T} is called the **tangible submonoid** of R . \mathcal{G} is called the **ghost ideal**.

We write a^ν for $\nu(a)$, for $a \in R$. We write $a \cong_\nu b$ if $a^\nu = b^\nu$, and say that a and b are ν -**equivalent**. Likewise we write $a \geq_\nu b$ (resp. $a >_\nu b$) if $a^\nu \geq b^\nu$ (resp. $a^\nu > b^\nu$).

Definition 2.5. A ν -**semifield**[†] is a ν -domain[†] for which the tangible submonoid \mathcal{T} is an Abelian group. A **supertropical semifield**[†] is a ν -semifield[†] $F = (F, \mathcal{T}, \nu, \mathcal{G})$ for which $F = \mathcal{T} \cup \mathcal{G}$ and \mathcal{G} is totally ordered, a special case of [12].

Example 2.6. Given a monoid M and an ordered group \mathcal{G} with an isomorphism $\nu : M \rightarrow \mathcal{G}$, we write a^ν for $\nu(a)$. The **standard supertropical monoid** R is the disjoint union $\mathcal{T} \cup \mathcal{G}$ where \mathcal{T} is taken to be M , made into a monoid by starting with the given products on M and \mathcal{G} , and defining ab^ν and $a^\nu b$ to be $(ab)^\nu$ for $a, b \in M$.

We extend ν to the **ghost map** $\nu : R \rightarrow \mathcal{G}$ by taking $\nu|_M = \nu$ and $\nu_\mathcal{G}$ to be the identity on \mathcal{G} . Thus, ν is a monoid projection.

We make R into a semiring[†], called the **standard supertropical semifield**[†], by defining

$$a + b = \begin{cases} a & \text{for } a >_\nu b; \\ b & \text{for } a <_\nu b; \\ a^\nu & \text{for } a \cong_\nu b. \end{cases}$$

R is never additively cancellative, since

$$a + a^\nu = a^\nu = a^\nu + a^\nu.$$

2.2.1. ν -Localization.

If $R = (R, \mathcal{T}, \mathcal{G}, \nu)$ is a ν -domain[†], then we call $\mathcal{T}^{-1}R$ the ν -**semifield**[†] of fractions $\text{Frac}_\nu R$ of R .

Lemma 2.7. $\text{Frac}_\nu R$ is a ν -semifield[†] in the obvious way.

Proof. Define $\nu(\frac{r}{s}) = \frac{r^\nu}{s^\nu}$. □

2.3. Kernels of Semiring[†].

The role of ideals is replaced here by kernels.

Definition 2.8. A **kernel** of a semiring[†] \mathcal{S} is a subgroup K which is **convex** in the sense that if $a, b \in K$ and $\alpha, \beta \in \mathcal{S}$ with $\alpha + \beta = \mathbf{1}_F$, then $\alpha a + \beta b \in K$.

Proposition 2.9. [23, Proposition 4.1.3] *If Ω is a congruence on a semifield[†] \mathcal{S} , then $K_\Omega = \{a \in \mathcal{S} : a \equiv 1\}$ is a kernel. Conversely, any kernel K of \mathcal{S} defines a congruence according to [7, Definition 3.1], i.e., $a \equiv b$ iff $\frac{a}{b} \equiv 1$. If \mathcal{S} is the semifield[†] of the lattice-ordered group G , then the semifield[†] \mathcal{S}/ρ_K is the semifield[†] of the lattice-ordered group G/K .*

Remark 2.10. [23, Remark 4.1.4]

- (i) [29, Corollary 1.1], [28, Property 2.4] Any kernel K is convex with respect to the order of Remark 2.3, in the sense that if $a \leq b \leq c$ with $a, c \in K$, then $b \in K$.
- (ii) [29, Proposition 2.3]. If $|a| \in K$, a kernel, then $a \in K$.
- (iii) [29] The product $K_1 K_2 = \{ab : a \in K_1, b \in K_2\}$ of two kernels is a kernel, in fact the smallest kernel containing $K_1 \cup K_2$.
- (iv) The intersection of kernels is a kernel. Thus, for any set $S \subset \mathcal{S}$ we can define the kernel $\langle S \rangle$ **generated by S** to be the intersection of all kernels containing S .
- (v) [29, Theorem 3.5]. Any kernel generated by a finite set $\{s_1, \dots, s_m\}$ is in fact generated by the single element $\sum_{i=1}^m (s_i + s_i^{-1})$.
- (vi) The kernel generated by $a \in \mathcal{S}$ is just the set of finite sums $\{\sum_i b_i a^i : b_i \in \mathcal{S}, \sum b_i = 1\}$.
- (vii) [7, Theorem 3.8]. If K is a kernel of a semifield[†] \mathcal{S} and the semifield[†] \mathcal{S}/K is idempotent, then K is a sub-semifield[†] of \mathcal{S} . (This is because for $a, b \in K$ the image of $a + b$ is $1K + 1K = 1K$.)
- (viii) Let K be a kernel of a semifield[†] \mathcal{S} . For every $a \in \mathcal{S}$, if $a^n \in K$ for some $n \in \mathbb{N}$ then $a \in K$.
- (ix) The kernel of a kernel is a kernel.

We also need the following generalization of (vi):

Proposition 2.11. [7, Proposition (3.13)] *Let \mathbb{S} be a semifield and let N be a (normal) subgroup of (\mathbb{S}, \cdot) . Then the smallest kernel containing N is*

$$(2.3) \quad \left\{ \sum_{i=1}^n s_i h_i : n \in \mathbb{N}, h_i \in N, s_i \in \mathbb{S} \text{ such that } \sum_{i=1}^n s_i = 1 \right\}.$$

Next, we recall [23, Theorem 4.1.6 ff.], which really is a special case of the basic lattice correspondence from universal algebra:

Theorem 2.12. *Let $\phi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ be a semifield[†] homomorphism. Then the following hold:*

- (1) *For any kernel L of \mathcal{S}_1 , $\phi(L)$ is a kernel of $\phi(\mathcal{S}_1)$.*
- (2) *For a kernel K of $\phi(\mathcal{S}_1)$, $\phi^{-1}(K)$ is a kernel of \mathcal{S}_1 . In particular, for any kernel L of \mathcal{S}_1 , $\phi^{-1}(\phi(L)) = KL$ is a kernel of \mathcal{S}_1 .*

In particular, $\phi^{-1}(1)$ is a kernel.

Corollary 2.13. *There is an injection $\mathbb{S}/(K_1 \cap K_2 \cap \cdots \cap K_t) \hookrightarrow \prod_{i=1}^t \mathbb{S}/K_i$, for any kernels K_i of a semifield[†] \mathbb{S} , induced by the map $f \mapsto (fK_i)$.*

We also have the isomorphism theorems.

Theorem 2.14. *Let \mathcal{S}, L be kernels of \mathcal{S} .*

- (1) *If \mathcal{U} is a sub-semifield[†] of \mathcal{S} , then $\mathcal{U} \cap K$ is a kernel of \mathcal{U} , and K a kernel of the sub-semifield[†] $\mathcal{U}K = \{u \cdot k : u \in \mathcal{U}, k \in K\}$ of \mathcal{S} , and one has the isomorphism*

$$\mathcal{U}/(\mathcal{U} \cap K) \cong \mathcal{U}K/K.$$

- (2) *$L \cap K$ is a kernel of L and K a kernel of LK , and the group isomorphism*

$$L/(L \cap K) \cong LK/K$$

is a semifield[†] isomorphism.

- (3) *If $L \subseteq K$, then K/L is a kernel of \mathcal{S}/L and one has the semifield[†] isomorphism*

$$\mathcal{S}/K \cong (\mathcal{S}/L)/(K/L).$$

Let L be a kernel of a semifield[†] \mathcal{S} . Every kernel of \mathcal{S}/L has the form K/L for some uniquely determined kernel $K \supseteq L$, yielding a lattice isomorphism

$$\{\text{Kernels of } \mathcal{S}/L\} \rightarrow \{\text{Kernels of } \mathcal{S} \text{ containing } L\}$$

given by $K/L \mapsto K$.

2.4. Principal kernels.

Here are more properties of kernels of semifields[†] in terms of their generators. \mathcal{S} always denotes an idempotent semifield[†].

Definition 2.15. For a subset S of \mathcal{S} , denote by $\langle S \rangle$ the smallest kernel in \mathcal{S} containing S , i.e., the intersection of all kernels in \mathcal{S} containing S . A kernel K is said to be **finitely generated** if $K = \langle S \rangle$ where S is a finite set. If $K = \langle a \rangle$ for some $a \in \mathcal{S}$, then K is called a **principal kernel**.

For convenience, we only consider kernels of polynomials with tangible coefficients; in [23] we treated the more general situation of arbitrary coefficients. We say that $f \in F(\Lambda)$ is **positive** if $f(\mathbf{a}) \geq 1$ for each $\mathbf{a} \in F^{(n)}$. Given $f = \frac{h}{g}$ for $h, g \in \mathcal{T}(\Lambda)$, we define $|f| = f + f^{-1}$. Clearly $|f|$ is positive, and $|f| = 1$ iff $f = 1$.

Lemma 2.16. [28, Property 2.3] *Let K be a kernel of an idempotent semifield[†] \mathcal{S} . Then for $a, b \in \mathcal{S}$,*

$$(2.4) \quad |a| \in K \quad \text{or} \quad |a| + b \in K \quad \Rightarrow \quad a \in K.$$

Proposition 2.17. [28, Proposition (3.1)]

$$(2.5) \quad \langle a \rangle = \{x \in \mathcal{S} : \exists n \in \mathbb{N} \text{ such that } a^{-n} \leq x \leq a^n\}.$$

Corollary 2.18. [23, Corollary 4.1.16] *For any $a \in \mathcal{S}$,*

$$\langle a \rangle = \{x \in \mathcal{S} : \exists n \in \mathbb{N} \text{ such that } |a|^{-n} \leq x \leq |a|^n\}.$$

Definition 2.19. A semifield[†] is said to be **finitely generated** if it is finitely generated as a kernel. If $\mathcal{S} = \langle a \rangle$ for some $a \in \mathcal{S}$, then \mathcal{S} is said to be a **principal semifield[†]**, with **generator** a .

Theorem 2.20. [23, Theorem 4.1.19] *If an archimedean idempotent semifield[†] F has a finite number of generators a_1, \dots, a_n , then F is a principal semifield[†], generated by $a = |a_1| + \cdots + |a_n|$.*

Note 2.21. [23] In view of Proposition 2.17, $\mathcal{S} = \langle \alpha \rangle$ for each $\alpha \neq \{1\}$.

Definition 2.22. By **sublattice** of the lattice of kernels, we mean a subset that is a lattice with respect to intersection and multiplication.

Corollary 2.23. [23, Corollary 4.1.26] *The set of principal kernels of an idempotent semifield[†] forms a sublattice of the lattice of kernels.*

Corollary 2.24. [23, Corollary 4.1.27] *For any generator a of a semifield[†] F , $F(\Lambda) = \langle a \rangle \prod_{i=1}^n \langle \lambda_i \rangle$, and $F(\Lambda)$ is a principal semifield[†] with generator $\sum_{i=1}^n |\lambda_i| + |a|$.*

2.5. ν -kernels.

Let us make this all supertropical.

Definition 2.25. A ν -congruence on a ν -domain[†] R is a congruence Ω for which $(a, b) \in \Omega$ iff $(a^\nu, b^\nu) \in \Omega$. We write $a_1 \equiv_\nu a_2$ when $a_1^\nu \equiv a_2^\nu$.

Remark 2.26. [23, Remark 4.1.27] For any congruence Ω of \mathcal{G} , $\nu^{-1}(\Omega) := \{(a, b) : a \cong_\nu b\}$ is a ν -congruence of R .

Any ν -congruence $\Omega = \{(a, b) : a, b \in R\}$ of R defines a congruence $\Omega^\nu = \{(a^\nu, b^\nu) : (a, b) \in \Omega\}$ of \mathcal{G} . Conversely, if Ω_ν is a congruence of \mathcal{G} , then $\nu^{-1}(\Omega_\nu)$ is a ν -congruence of R .

Similarly, we have:

Definition 2.27. A ν -kernel of a ν -semifield[†] \mathbb{S} is a subgroup K which is ν -convex in the sense that if $a, b \in K$ and $\alpha, \beta \in \mathbb{S}$ with $\alpha + \beta \cong_\nu \mathbf{1}_F$, then $\alpha a + \beta b \in K$.

Remark 2.28. Any ν -kernel \mathcal{K} of a ν -semifield[†] \mathbb{S} defines a kernel \mathcal{K}^ν of \mathcal{G} . Conversely, if \mathcal{K} is a kernel of \mathcal{G} , then $\nu^{-1}(\mathcal{K})$ is a ν -kernel of \mathbb{S} .

If $\mathcal{K} = \mathcal{K}_\Omega$, then $\nu^{-1}(\mathcal{K}) = \mathcal{K}_{\nu^{-1}(\Omega)}$.

We recall the monoid automorphism $(*)$ of order 2 of [23, Remark 3.3.1] given by

$$a^* = a^{-1}, \quad (a^\nu)^* = (a^{-1})^\nu, \quad a \in \mathcal{T}.$$

We also define the lattice supremum $a \wedge b = a + b$ and

$$(2.6) \quad a \wedge b = (a^* + b^*)^*,$$

which is the lattice infimum.

Remark 2.29. [23, Remark 4.2.5]

- (i) Given a ν -congruence Ω on a ν -semifield[†] \mathbb{S} , we define $K_\Omega = \{a \in \mathbb{S} : a \equiv_\nu 1\}$. Conversely, given a ν -kernel K of \mathbb{S} , we define the ν -congruence Ω on \mathbb{S} by $a \equiv b$ iff $ab^* \equiv_\nu \mathbf{1}_F$.
- (ii) Any ν -kernel K is ν -convex, in the sense that if $a \leq_\nu b \leq_\nu c$ with $a, c \in K$, then $b \in K$.
- (iii) If $|a| \in K$, a ν -kernel, then $a \in K$.
- (iv) The product of two ν -kernels is a ν -kernel.
- (v) The intersection of ν -kernels is a ν -kernel.
- (vi) Any ν -kernel generated by a finite set $\{s_1, \dots, s_m\}$ is generated by the single element $\sum_{i=1}^m (|s_i|)$.
- (vii) The ν -kernel generated by $a \in \mathbb{S}$ is just the set of finite sums $\{\sum_i b_i a^i : b_i \in \mathbb{S}, \sum b_i \cong_\nu 1\}$.
- (viii) If K is a ν -kernel of a ν -semifield[†] \mathbb{S} and the ν -semifield[†] \mathbb{S}/K is ν -idempotent, then K is a sub- ν -semifield[†] of \mathbb{S} . (This is because for $a, b \in K$ the image of $a + b$ is ν -equivalent to $1K + 1K = 1K$.)

In view of (v) for any set $S \subset \mathbb{S}$ we can define the ν -kernel $\langle S \rangle$ **generated by S** to be the intersection of all ν -kernels containing S . In what follows, we only consider ν -kernels generated by tangible elements, in order to avoid “ghost kernels” and obtain the following observation.

Proposition 2.30. *For any $\gamma_1, \dots, \gamma_n \in F$ the kernel $\langle \frac{\lambda_1}{\gamma_1}, \dots, \frac{\lambda_n}{\gamma_n} \rangle$ is a maximal kernel of $F(\Lambda)$.*

Proof. The quotient is isomorphic to F , which is simple. □

2.6. The kernel $\langle F \rangle$.

$\langle F \rangle$ denotes the kernel of $F(\Lambda)$ generated by any element $\alpha \neq 1$ of F . This kernel plays a special role in the theory, as seen in [23].

Corollary 2.31. $F(\Lambda) = F \cdot L_{(\alpha_1, \dots, \alpha_n)} = \langle F \rangle \cdot L_{(\alpha_1, \dots, \alpha_n)}$.

Lemma 2.32. *If K is a maximal kernel of $\langle F \rangle$, then*

$$K \in \Omega \left(\left\langle \frac{\lambda_1}{\alpha_1}, \dots, \frac{\lambda_n}{\alpha_n} \right\rangle \right)$$

for suitable $\alpha_1, \dots, \alpha_n \in F$.

Proof. Denote $L_a = (|\frac{\lambda_1}{\alpha_1}| + \dots + |\frac{\lambda_n}{\alpha_n}|) \wedge |\alpha|$ with $\alpha \neq 1$, for $a = (\alpha_1, \dots, \alpha_n)$. We may assume that $1_{\text{loc}}(K) \neq \emptyset$, since the kernel corresponding to the empty set is $\langle F \rangle$ itself. If $a \in 1_{\text{loc}}(K)$, then $\langle L_a \rangle \supseteq K$ since $1_{\text{loc}}(L_a) = \{a\} \subseteq 1_{\text{loc}}(K)$. Thus, the maximality of K implies that $K = \langle L_a \rangle$. \square

2.7. The Zariski correspondence for ν -kernels.

Definition 2.33. A **kernel root** of $f \in \text{Fun}(F^{(n)}, F)$ is an element $\mathbf{a} \in F^{(n)}$ such that $f(\mathbf{a}) \cong_\nu 1_F$.

For $S \subseteq F(\Lambda)$, define

$$(2.7) \quad 1_{\text{loc}}(S) = \{\mathbf{a} \in F^{(n)} : f(\mathbf{a}) \cong_\nu 1, \forall f \in S\}.$$

We write $1_{\text{loc}}(f)$ for $1_{\text{loc}}(\{f\})$.

Definition 2.34. A subset $Z \subset F^{(n)}$ is said to be a 1^ν -**set** if there exists a subset $S \subset F(\Lambda)$ such that $Z = 1_{\text{loc}}(S)$.

2.8. The coordinate ν -semifield † of a 1^ν -set.

Definition 2.35. For $X \subset F^{(n)}$, The **coordinate ν -semifield †** $F(X)$ of a 1^ν -set X is the set of restriction of the rational functions $F(\Lambda)$ to X .

$$\phi_X : F(\Lambda) \rightarrow F(X)$$

denotes the restriction map $h \mapsto h|_X$.

The tropical significance comes from:

Theorem 2.36. [23, Theorem 7.1.7] *The correspondences $f \mapsto \hat{f}$ and $h \mapsto \underline{h}$ induces a 1:1 correspondence between corner hypersurfaces and 1^ν -sets of corner internal rational functions.*

Proposition 2.37. [23, Proposition 5.4.2] ϕ_X is an onto semifield † homomorphism.

Proposition 2.38. [23, Proposition 5.4.3] $F(X)$ is a ν -domain † , isomorphic to $F(\Lambda)/\text{Kern}(X)$.

Thus, chains of kernels of $F(\Lambda)$ give us an algebraic tropical notion of dimension, and its determination is the subject of this paper.

2.9. The Jordan-Hölder theorem.

Our main goal is to find a Jordan-Hölder theorem for kernels. But there are too many kernels for a viable theory in general, as discussed in [1]. If we limit our set of kernels to a sublattice of kernels, one can use the Schreier refinement theorem [22] to obtain a version of the Jordan-Hölder Theorem.

Definition 2.39. $\mathcal{L}(S)$ denotes the lattice of ν -kernels of a ν -semifield † S .

Θ is a **natural map** if for each ν -semifield † S , there is a lattice homomorphism $\Theta_S : \mathcal{L}(S) \rightarrow \mathcal{L}(S)$ such that $K \mapsto \Theta_S(K)$ is a homomorphism of kernels. We write $\Theta(S)$ for $\Theta_S(\mathcal{L}(S))$, and call the kernels in Θ_S **Θ -kernels**. (We delete S when it is unambiguous.)

A $\Theta(S)$ -**simple** kernel is a minimal Θ -kernel $\neq \{1\}$. A $\Theta(S)$ -**composition series** $\mathcal{C}(K, L)$ in $\Theta(S)$ from a kernel K to a subkernel L is a chain

$$K = K_0 \supset K_1 \supset \dots \supset K_t = L$$

in Θ such that each factor is Θ -simple.

By Theorem 2.12 $\mathcal{C}(K, L)$ is equivalent to the $\Theta(S/L)$ -composition series

$$K/L \supset K_1/L \supset \dots \supset K_t/L = 0$$

of K/L .

Given a Θ -kernel K , we define its **composition length** $\ell(K)$ to be the length of a Θ -composition series for K (presuming K has one). By definition, $\{1\}$ is the only Θ -kernel of composition length 0. A nonzero Θ -kernel K is simple iff $\ell(K) = 1$. The next theorem is a standard lattice-theoretic result of Schreier and Zassenhaus, yielding the Jordan-Hölder Theorem, cf. [25, Theorem 3.11, Schreier-Jordan-Hölder Theorem].

Theorem 2.40. *Suppose K has a composition series*

$$K = K_0 \supset K_1 \supset \dots \supset K_t = 0,$$

which we denote as \mathcal{C} . Then:

(i) *Any arbitrary finite chain of subkernels*

$$K = N_0 \supset N_1 \supset \dots \supset N_k \supset 0$$

(denoted as \mathcal{D}), can be refined to a composition series equivalent to \mathcal{C} . In particular, $k \leq t$.

(ii) Any two composition series of K are equivalent.

(iii) $\ell(K) = \ell(N) + \ell(K/N)$ for every subkernel N of K . In particular, every subkernel and every homomorphic image of a kernel with composition series has a composition series.

2.10. The HO-decomposition.

Definition 2.41. For any kernel K of $F(\Lambda)$, define the equivalence relation

$$(2.8) \quad f \sim_K f' \quad \text{if and only if} \quad \langle f \rangle \cap K = \langle f' \rangle \cap K$$

as kernels of $F(\Lambda)$. The equivalence classes are

$$[f] = \{f' : f' \text{ is a generator of } \langle f \rangle \cap K\}.$$

Our interest is in $K = \langle F \rangle$.

Definition 2.42. An \mathcal{L} -monomial is a non-constant Laurent monomial $f \in F(\lambda_1, \dots, \lambda_n)$; i.e., $f = \frac{h}{g}$ with $h, g \in F[\lambda_1, \dots, \lambda_n]$ non-proportional monomials.

A rational function $f \in F(\lambda_1, \dots, \lambda_n)$ is called a **hyperspace-fraction**, or HS-fraction, if $f \sim_{\langle F \rangle} \sum_{i=1}^t |f_i|$ where the f_i are non-proportional \mathcal{L} -monomials.

Definition 2.43. A 1^ν -set in $F^{(n)}$ is a **hyperplane 1^ν -set** (HP- 1^ν -set for short) if it is defined by an \mathcal{L} -monomial. A 1^ν -set in $F^{(n)}$ is a **hyperspace-fraction 1^ν -set** (HS- 1^ν -set for short) if it is defined by an HS-fraction.

Proposition 2.44. [23, Corollary 9.1.10] *A 1^ν -set is an HS- 1^ν -set if and only if it is an intersection of HP- 1^ν -sets.*

$\mathcal{P}(K)$ denotes the lattice of principal subkernels of a kernel K .

Definition 2.45. $\text{HP}(K)$ denotes the family of \mathcal{L} -monomials in a kernel K . A **hyperplane kernel**, or **HP-kernel**, for short, is a principal kernel of $F(\lambda_1, \dots, \lambda_n)$ generated by an \mathcal{L} -monomial.

A **hyperspace-fraction kernel**, or **HS-kernel**, for short, is a principal kernel of $F(\lambda_1, \dots, \lambda_n)$ generated by a hyperspace fraction.

Definition 2.46. $\Omega(F(\Lambda))$ is the lattice of kernels finitely generated by HP-kernels of $F(\Lambda)$, i.e., every element $\langle f \rangle \in \Omega(F(\Lambda))$ is obtained via finite intersections and products of HP-kernels.

Proposition 2.47. [23, Proposition (9.1.7)] *Any principal HS-kernel is a product of distinct HP-kernels, and thus is in $\Omega(F(\Lambda))$.*

Lemma 2.48. [23, Lema 9.1.11] *Let $\langle f \rangle$ be an HP-kernel, with F divisible. If $w \in \langle f \rangle$ is an \mathcal{L} -monomial, then $w^s = f^k$ for some $s, k \in \mathbb{Z} \setminus \{0\}$.*

Before refining this description, we recall [23, Construction 2.6.1], to fix notation.

Construction 2.49. Take a rational function $f \in F(\lambda_1, \dots, \lambda_n)$ for which $1_{\text{loc}}(f) \neq \emptyset$. Replacing f by $|f|$, we may assume that $f \geq_\nu 1$. Write $f = \frac{h}{g} = \frac{\sum_{i=1}^k h_i}{\sum_{j=1}^m g_j}$ where h_i and g_j are monomials in $F[\lambda_1, \dots, \lambda_n]$. For each $\mathbf{a} \in 1_{\text{loc}}(f)$, let

$$H_{\mathbf{a}} \subseteq H = \{h_i : 1 \leq i \leq k\}; \quad G_{\mathbf{a}} \subseteq G = \{g_j : 1 \leq j \leq m\}$$

be the sets of dominant monomials at \mathbf{a} ; thus, $h_i(\mathbf{a}) = g_j(\mathbf{a})$ for any $h_i \in H_{\mathbf{a}}$ and $g_j \in G_{\mathbf{a}}$. Let $H_{\mathbf{a}}^c = H \setminus H_{\mathbf{a}}$ and $G_{\mathbf{a}}^c = G \setminus G_{\mathbf{a}}$. Then, for any $h' \in H_{\mathbf{a}}$ and $h'' \in H_{\mathbf{a}}^c$, $h'(\mathbf{a}) + h''(\mathbf{a}) = h'(\mathbf{a})$, or, equivalently, $1 + \frac{h''(\mathbf{a})}{h'(\mathbf{a})} = 1$. Similarly, for any $g' \in G_{\mathbf{a}}$ and $g'' \in G_{\mathbf{a}}^c$, $g'(\mathbf{a}) + g''(\mathbf{a}) = g'(\mathbf{a})$ or, equivalently, $1 + \frac{g''(\mathbf{a})}{g'(\mathbf{a})} = 1$.

Thus for any such \mathbf{a} we obtain the relations

$$(2.9) \quad \frac{h'}{g'} = 1, \quad \forall h' \in H_{\mathbf{a}}, g' \in G_{\mathbf{a}},$$

$$(2.10) \quad 1 + \frac{h''}{h'} = 1; \quad 1 + \frac{g''}{g'} = 1, \quad \forall h' \in H_{\mathbf{a}}, h'' \in H_{\mathbf{a}}^c, g' \in G_{\mathbf{a}}, g'' \in G_{\mathbf{a}}^c.$$

As \mathbf{a} runs over $1_{\text{loc}}(f)$, there are only finitely many possibilities for $H_{\mathbf{a}}$ and $G_{\mathbf{a}}$ and thus for the relations in (2.9) and (2.10); we denote these as $(\theta_1(i), \theta_2(i))$, $i = 1, \dots, q$.

In other words, for any $1 \leq i \leq q$, the pair $(\theta_1(i), \theta_2(i))$ corresponds to a kernel K_i generated by the corresponding elements

$$\frac{h'}{g'}, \left(1 + \frac{h''}{h'}\right), \text{ and } \left(1 + \frac{g''}{g'}\right),$$

where $\{\frac{h'}{g'} = 1\} \in \theta_1$ and $\{1 + \frac{g''}{g'} = 1\}, \{1 + \frac{h''}{h'} = 1\} \in \theta_2$.

Reversing the argument, every point satisfying one of these q sets of relations is in $1_{\text{loc}}(f)$. Hence,

$$(2.11) \quad \begin{aligned} 1_{\text{loc}}(\langle f \rangle \cap \langle F \rangle) &= 1_{\text{loc}}(f) = \bigcup_{i=1}^q 1_{\text{loc}}(K_i) = \bigcup_{i=1}^q 1_{\text{loc}}(K_i \cap \langle F \rangle) \\ &= 1_{\text{loc}}\left(\bigcap_{i=1}^q (K_i \cap \langle F \rangle)\right), \end{aligned}$$

Hence $\langle f \rangle \cap \langle F \rangle = \bigcap_{i=1}^q K_i \cap \langle F \rangle$, since $\langle f \rangle \cap \langle F \rangle, \bigcap_{i=1}^q K_i \cap \langle F \rangle \in \mathcal{P}(\langle F \rangle)$. $\bigcap_{i=1}^q K_i$ provides a local description of f in a neighborhood of its 1^ν -set.

Let us view this construction globally. We used the 1^ν -set of $\langle f \rangle$ to construct $\bigcap_{i=1}^q K_i$. Adjoining various points \mathbf{a} in $F^{(n)}$ might add some regions, complementary to the regions defined by (2.10) in $\theta_2(i)$ for $i = 1, \dots, q$, over which $\frac{h'}{g'} \neq 1, \forall h' \in H_{\mathbf{a}}, \forall g' \in G_{\mathbf{a}}$ for each \mathbf{a} , i.e., regions over which the dominating monomials never agree. Continuing the construction above using $\mathbf{a} \in F^{(n)} \setminus 1_{\text{loc}}(f)$ similarly produces a finite collection of, say $t \in \mathbb{Z}_{>0}$, kernels generated by elements from (2.10) and their complementary order fractions and by elements of the form (2.9) (where now $\frac{h'}{g'} \neq 1$ over the region considered). Any principal kernel $N_j = \langle q_j \rangle$, $1 \leq j \leq t$, of this complementary set of kernels has the property that $1_{\text{loc}}(N_j) = \emptyset$, and thus by Corollary ??, N_j is bounded from below. As there are finitely many such kernels there exists small enough $\gamma >_\nu 1$ in \mathcal{T} for which $|q_j| \wedge \gamma = \gamma$ for $j = 1, \dots, t$. Thus $\bigcap_{j=1}^t N_j$ is bounded from below and thus $\bigcap_{j=1}^t N_j \supseteq \langle F \rangle$ by Remark ??.

Piecing this together with (2.11) yields f over all of $F^{(n)}$, so we have

$$(2.12) \quad \langle f \rangle = \bigcap_{i=1}^q K_i \cap \bigcap_{j=1}^t N_j.$$

$$\text{So, } \langle f \rangle \cap \langle F \rangle = \bigcap_{i=1}^q K_i \cap \bigcap_{j=1}^t N_j \cap \langle F \rangle = \bigcap_{i=1}^q K_i \cap \langle F \rangle.$$

In this way, we see that intersecting a principal kernel $\langle f \rangle$ with $\langle F \rangle$ ‘chops off’ all of the bounded from below kernels in (2.12) (the N_j ’s given above). This eliminates ambiguity in the kernel corresponding to $1_{\text{loc}}(f)$. Finally we note that if $1_{\text{loc}}(f) = \emptyset$, then $\langle f \rangle = \bigcap_{j=1}^t N_j$ for appropriate kernels N_j and $\langle f \rangle \cap \langle F \rangle = \langle F \rangle$.

Remark 2.50.

- (i) If K_1 and K_2 are such that $K_1 K_2 \cap F = \{1\}$ (i.e., $1_{\text{loc}}(K_1) \cap 1_{\text{loc}}(K_2) \neq \emptyset$), then the sets of \mathcal{L} -monomials θ_1 of K_1 and of K_2 are not the same (although one may contain the other), for otherwise together they would yield a single kernel via Construction 2.49.
- (ii) The kernels K_i , being finitely generated, are in fact principal, so we can write $K_i = \langle k_i \rangle$ for rational functions k_1, \dots, k_q . Let $\langle f \rangle \cap \langle F \rangle = \bigcap_{i=1}^q (K_i \cap \langle F \rangle) = \bigcap_{i=1}^q \langle |k_i| \wedge |\alpha| \rangle = \bigwedge_{i=1}^q \langle |k_i| \wedge |\alpha| \rangle$ with $\alpha \in F \setminus \{1\}$. By [23, Theorem 8.5.3], for any generator f' of $\langle f \rangle \cap \langle F \rangle$ we have $|f'| = \bigwedge_{i=1}^q |k'_i|$ with $k'_i \sim_{\langle F \rangle} |k_i| \wedge |\alpha|$ for every $i = 1, \dots, q$. In particular, $1_{\text{loc}}(k'_i) = 1_{\text{loc}}(|k_i| \wedge |\alpha|) = 1_{\text{loc}}(k_i)$. Thus the kernels K_i are independent of the choice of generator f , being defined by the components $1_{\text{loc}}(k_i)$ of $1_{\text{loc}}(f)$.

Two instances of Construction 2.49 are given in [23, Examples 2.6.3, 2.6.4].

Definition 2.51. A rational function $g \in F(\Lambda)$ is **bounded from below** if there exists some $\alpha >_\nu 1$ in F such that $|g| \geq_\nu \alpha$.

An important instance: the \mathcal{L} -**binomial** o defined by an \mathcal{L} -monomial f is the rational function $1 + f$. The **complementary \mathcal{L} -binomial** o^c of o is $1 + f^{-1}$. By definition $(\mathcal{O}^c)^c = \mathcal{O}$. The **order kernel** of the semifield[†] $F(\lambda_1, \dots, \lambda_n)$ defined by f is the principal kernel $\mathcal{O} = \langle o \rangle$ for the \mathcal{L} -binomial $o = 1 + f$. The **complementary order kernel** \mathcal{O}^c of \mathcal{O} is $\langle o^c \rangle$.

A rational function $f \in F(\lambda_1, \dots, \lambda_n)$ is said to be a **region fraction** if $1_{\text{loc}}(f)$ contains some nonempty open interval. A **region kernel** is a principal kernel generated by a region fraction.

Lemma 2.52. [23, Lemma 9.1.16] $f \sim_{\langle F \rangle} \sum_{i=1}^t |o_i|$ is a region fraction iff, writing $o_i = 1 + f_i$ for \mathcal{L} -monomials f_i , we have $f_i \not\sim_{\nu} f_j^{\pm 1}$ for every $i \neq j$.

Definition 2.53. A rational function $f \in F(\lambda_1, \dots, \lambda_n)$ is an **HO-fraction** if it is the sum of an HS-fraction f' and a region fraction o_f . (In particular, any HS-kernel or any region kernel is an HO-kernel.)

A principal kernel $K \in \mathcal{P}(F(\lambda_1, \dots, \lambda_n))$ is said to be an **HO-kernel** if it is generated by an HO-fraction.

Lemma 2.54. [23, Lemma 2.6.7] A principal kernel K is an HO-kernel if and only if $K = LR$ where L is an HS-kernel and R is a region kernel.

Theorem 2.55. [23, Theorem 2.6.8] Every principal kernel $\langle f \rangle$ of $F(\lambda_1, \dots, \lambda_n)$ can be written as the intersection of finitely many principal kernels

$$\{K_i : i = 1, \dots, q\} \text{ and } \{N_j : j = 1, \dots, m\},$$

whereas each K_i is the product of an HS-kernel and a region kernel

$$(2.13) \quad K_i = L_i R_i = \prod_{j=1}^{t_i} L_{i,j} \prod_{k=1}^{k_i} \wr_{i,k}$$

while each N_j is a product of bounded from below kernels and (complementary) region kernels. For $\langle f \rangle \in \mathcal{P}(\langle F \rangle)$, the N_j can be replaced by $\langle F \rangle$ without affecting $\langle f \rangle$.

3. CONVEXITY DEGREE AND HYPERDIMENSION

Let $\langle f \rangle \subseteq \langle F \rangle$ be a principal kernel and let $\langle f \rangle = \bigcap_{i=1}^s K_i$, where

$$K_i = (L_i \cdot R_i) \cap \langle F \rangle = (L_i \cap \langle F \rangle) \cdot (R_i \cap \langle F \rangle) = L'_i \cdot R'_i$$

is its (full) HO-decomposition; i.e., for each $1 \leq i \leq s$, $R_i \in \mathcal{P}(F)$ is a region kernel and $L_i \in \mathcal{P}(F)$ is either an HS-kernel or bounded from below (in which case $L'_i = \langle F \rangle$). Then by Corollary 2.13, we have the subdirect decomposition

$$\langle F \rangle / \langle f \rangle \hookrightarrow \prod_{i=1}^t \langle F \rangle / K_i = \prod_{i=1}^t (\langle F \rangle / L'_i \cdot R'_i)$$

where $t \leq s$ is the number of kernels K_i for which $L'_i \neq \langle F \rangle$ (for otherwise $\langle F \rangle / K_i = \{1\}$ and can be omitted from the subdirect product).

Example 3.1. Consider the principal kernel $\langle \lambda_1 \rangle \in \mathcal{P}(F(\lambda_1, \lambda_2))$. For $\alpha \in F$ such that $\alpha > 1$, we have the following infinite strictly descending chain of principal kernels

$$\begin{aligned} \langle \lambda_1 \rangle &\supset \langle |\lambda_1| + |\lambda_2 + 1| \rangle \supset \langle |\lambda_1| + |\alpha^{-1} \lambda_2 + 1| \rangle \supset \langle |\lambda_1| + |\alpha^{-2} \lambda_2 + 1| \rangle \supset \dots \\ &\supset \langle |\lambda_1| + |\alpha^{-k} \lambda_2 + 1| \rangle \supset \dots \end{aligned}$$

and the strictly ascending chain of 1^ν -sets corresponding to it.

$$1\text{-set}(\lambda_1) \subset 1\text{-set}(|\lambda_1| + |\lambda_2 + 1|) \subset \dots \subset 1\text{-set}(|\lambda_1| + |\alpha^{-k} \lambda_2 + 1|) \subset \dots =$$

$$1\text{-set}(\lambda_1) \subset 1\text{-set}(\lambda_1) \cap 1\text{-set}(\lambda_2 + 1) \subset \dots \subset 1\text{-set}(\lambda_1) \cap 1\text{-set}(\alpha^{-k} \lambda_2 + 1) \subset \dots$$

Example 3.2. Again, consider the principal kernel $\langle x \rangle \in \mathcal{P}(F(x, y))$. Then

$$\langle x \rangle = \langle |x| + (|y + 1| \wedge |\frac{1}{y} + 1|) \rangle = \langle (|x| + |y + 1|) \wedge (|x| + |\frac{1}{y} + 1|) \rangle = \langle |x| + |y + 1| \rangle \cap \langle |x| + |\frac{1}{y} + 1| \rangle.$$

So, we have the nontrivial decomposition of $1\text{-set}(x)$ as $1\text{-set}(|x| + |y + 1|) \cup 1\text{-set}(|x| + |\frac{1}{y} + 1|)$ (note that $1\text{-set}(|x| + |y + 1|) = 1\text{-set}(x) \cap 1\text{-set}(y + 1)$, and furthermore $1\text{-set}(|x| + |\frac{1}{y} + 1|) = 1\text{-set}(x) \cap 1\text{-set}(\frac{1}{y} + 1)$). In a similar way, using complementary order kernels, one can show that every principal kernel can be nontrivially decomposed to a pair of principal kernels.

3.1. Reducible kernels.

Examples 3.1 and 3.2 demonstrate that the lattice of principal kernels $\mathcal{P}(F(\Lambda))$ (resp. $\mathcal{P}(\langle F \rangle)$) is too rich to define reducibility or finite dimension. (See [1] for a discussion of infinite dimension.) Moreover, these examples suggest that this richness is caused by order kernels. This motivates us to consider Θ -reducibility for a suitable sublattice of kernels $\Theta \subset \mathcal{P}(F(\Lambda))$ (resp. $\Theta \subset \mathcal{P}(\langle F \rangle)$).

There are various families of kernels that could be utilized to define the notions of reducibility, dimensionality, and so forth. We take Θ to be the sublattice generated by HP-kernels, because of its connection to the (local) dimension of the linear spaces (in logarithmic scale) defined by the 1^ν -set corresponding to a kernel. Namely, HP-kernels, and more generally HS-kernels, define affine subspaces of $F^{(n)}$ (see [23, §9.2]). We work with Definition 2.46.

Definition 3.3. A kernel $\langle f \rangle \in \Omega(F(\Lambda))$ is **reducible** if there are $\langle g \rangle, \langle h \rangle \in \Omega(F(\Lambda))$ for which $\langle g \rangle, \langle h \rangle \not\subseteq \langle f \rangle$ but $\langle g \rangle \cap \langle h \rangle \subseteq \langle f \rangle$.

Lemma 3.4. $\langle f \rangle$ is reducible iff $\langle f \rangle = \langle g \rangle \cap \langle h \rangle$ where $\langle f \rangle \neq \langle g \rangle$ and $\langle f \rangle \neq \langle h \rangle$.

Proof. Assume $\langle f \rangle$ admits the stated condition. If $\langle f \rangle \supseteq \langle g \rangle \cap \langle h \rangle$, then $\langle f \rangle = \langle f \rangle \cdot \langle f \rangle = (\langle g \rangle \cdot \langle f \rangle) \cap (\langle h \rangle \cdot \langle f \rangle)$. Thus $\langle f \rangle = \langle g \rangle \cdot \langle f \rangle$ or $\langle f \rangle = \langle h \rangle \cdot \langle f \rangle$, implying $\langle f \rangle \supseteq \langle g \rangle$ or $\langle f \rangle \supseteq \langle h \rangle$. The converse is obvious. \square

Lemma 3.5. Let $\langle f \rangle$ be an HP-kernel. Then for any HP-kernels $\langle g \rangle$ and $\langle h \rangle$ such that $\langle f \rangle = \langle g \rangle \cap \langle h \rangle$ either $\langle f \rangle = \langle g \rangle$ or $\langle f \rangle = \langle h \rangle$. In other words, every HP-kernel is irreducible.

Proof. If $\langle f \rangle = \langle g \rangle \cap \langle h \rangle$ then $\langle f \rangle \subseteq \langle g \rangle$ thus $f \in \langle g \rangle$. As both f and g are \mathcal{L} -monomials (up to equivalence), Lemma 2.48 yields $\langle f \rangle = \langle g \rangle$, which in turn, by Lemma 3.4, implies that $\langle f \rangle$ is irreducible. \square

Corollary 3.6. Any HS-kernel $\langle f \rangle$ is irreducible.

Proof. If $\langle f \rangle = \langle g \rangle \cap \langle h \rangle$ for HP-kernels $\langle g \rangle$ and $\langle h \rangle$, then $\langle f \rangle \subseteq \langle g \rangle$. But $\langle f \rangle$ is a product $\langle f_1 \rangle \cdots \langle f_t \rangle$ of finitely many HP-kernels. For each $1 \leq j \leq t$, $\langle f_j \rangle \subseteq \langle g \rangle$ yielding $\langle g \rangle = \langle f_j \rangle$ by Lemma 2.48, and so $\langle f \rangle = \langle g \rangle$. \square

Corollary 3.7. If $\langle f \rangle = \langle g \rangle \cap \langle h \rangle$ for HS-kernels $\langle g \rangle$ and $\langle h \rangle$, then either $\langle f \rangle = \langle g \rangle$ or $\langle f \rangle = \langle h \rangle$.

Proof. Otherwise, since $\langle g \rangle$ and $\langle h \rangle$ are finite products of HP-kernels, there are HP-kernel $\langle g' \rangle \subseteq \langle g \rangle$ and $\langle h' \rangle \subseteq \langle h \rangle$ such that $\langle g' \rangle \not\subseteq \langle f \rangle$ and $\langle h' \rangle \not\subseteq \langle f \rangle$. But $\langle g' \rangle \cap \langle h' \rangle \subseteq \langle g \rangle \cap \langle h \rangle = \langle f \rangle$, implying $\langle f \rangle$ is reducible, contradicting Corollary 3.6. \square

Proposition 3.8. The irreducible kernels in the lattice generated by HP-kernels are precisely the HS-kernels.

Proof. This follows from Corollary 3.7, since all proper intersections in the lattice generated by HP-kernels are reducible. (Note that HP-kernels are also HS-kernels.) \square

Corollary 3.9. $\text{HSpec}(F(\Lambda))$ is the family of HS-kernels in $\Omega(F(\Lambda))$, which is precisely the family of HS-fractions of $F(\Lambda)$.

Definition 3.10. The **hyperspace spectrum** of $F(\Lambda)$, denoted $\text{HSpec}(F(\Lambda))$, is the family of irreducible kernels in $\Omega(F(\Lambda))$.

Definition 3.11. A chain $P_0 \subset P_1 \subset \cdots \subset P_t$ in $\text{HSpec}(F(\Lambda))$ of HS-kernels of $F(\Lambda)$ is said to have **length** t . An HS-kernel P has **height** t (denoted $\text{hgt}(P) = t$) if there is a chain of length t in $\text{HSpec}(F(\Lambda))$ terminating at P , but no chain of length $t + 1$ terminates at P .

Remark 3.12. Let L be a kernel in $\mathcal{P}(F(\Lambda))$. Consider the canonical homomorphism $\phi_L : F(\Lambda) \rightarrow F(\Lambda)/L$. Since the image of a principal kernel is generated by the image of any of its generators, $\phi_L(\langle f \rangle) = \langle \phi_L(f) \rangle$ for any HP-kernel $\langle f \rangle$. Choosing f to be an \mathcal{L} -monomial, $\langle \phi_L(f) \rangle$ is a nontrivial HP-kernel in $F(\Lambda)/L$ if and only if $\phi_L(f) \notin F$. Thus, the set of HP-kernels of $F(\Lambda)$ mapped to HP-kernels of $F(\Lambda)/L$ is

$$(3.1) \quad \{ \langle g \rangle : \langle g \rangle \cdot \langle F \rangle \supseteq \phi_L^{-1}(\langle F \rangle) = L \cdot \langle F \rangle \}.$$

As ϕ_L is an F -homomorphism, it respects \vee, \wedge and $|\cdot|$, and thus $\phi_L((\Omega(F(\Lambda)), \cap, \cdot)) = (\Omega(F(\Lambda)/L), \cap, \cdot)$. In fact Theorem 2.12 yields a correspondence identifying $\text{HSpec}(F(\Lambda)/L)$ with the subset of $\text{HSpec}(F(\Lambda))$ which consists of all HS-kernels P of $F(\Lambda)$ such that $P \cdot \langle F \rangle \supseteq L \cdot \langle F \rangle$.

Lemma 3.13. The above correspondence extends to a correspondence identifying $\Omega(F(\Lambda)/L)$ with the subset (3.1) of $\Omega(F(\Lambda))$. Under this correspondence, the maximal HS-kernels of $F(\Lambda)/L$ correspond to maximal HS-kernels of $F(\Lambda)$, and reducible kernels of $F(\Lambda)/L$ correspond to reducible kernels of $F(\Lambda)$.

Proof. The latter assertion is obvious since \wedge is preserved under homomorphisms. For the first assertion, $(F(\Lambda)/L)/(P/L) \cong F(\Lambda)/P$ by Theorem 2.14, so simplicity of the quotients is preserved. Hence, so is maximality of P/L and P . \square

Definition 3.14. The Hyperdimension of $F(\Lambda)$, written $\text{Hdim } F(\Lambda)$ (if it exists), is the maximal height of the HS-kernels in $F(\Lambda)$.

3.2. Decompositions.

Let us garner some information about reducible kernels from rational functions. Suppose $f \in F(\Lambda)$. We write $f = \sum_{i=1}^k f_i$ where each f_i is of the form $g_i h_i^*$ with $g_i, h_i \in F[\Lambda]$ and g_i a monomial. (This is $\frac{g_i}{h_i}$ when h_i is tangible.) We also assume that this sum is **irredundant** in the sense that we cannot remove any of the summands and still get f . If each time the value 1 is attained by one of the terms f_i in this expansion and all other terms attain values ≤ 1 , then $\tilde{f} = \bigwedge_{i=1}^k |f_i|$ defines the same 1^ν -set as f . Moreover, if $f \in \langle F \rangle$ then $\tilde{f} \wedge |\alpha| \in \langle F \rangle$, for $\alpha \in F \setminus \{1\}$ is also a generator of $\langle f \rangle$. The reason we take $\tilde{f} \wedge |\alpha|$ is that we have no guarantee that each of the f_i 's in the above expansion is bounded.

We can generalize this idea as follows:

We call $f \in F(\Lambda)$ **reducible** if we can write $f = \sum_{i=1}^k f_i$ as above, such that for every $1 \leq i \leq k$ the following condition holds:

$$f_i(\mathbf{a}) \cong_\nu 1 \Rightarrow f_j(\mathbf{a}) \leq_\nu 1, \forall j \neq i.$$

Definition 3.15. Let $f \in F(\Lambda)$. A Θ -decomposition of f is an expression of the form

$$(3.2) \quad |f| = |u| \wedge |v|$$

with u, v Θ -elements in $F(\Lambda)$.

The decomposition (3.2) is said to be trivial if $f \sim_{\langle F \rangle} u$ or $f \sim_{\langle F \rangle} v$ (equivalently $|f| \sim_{\langle F \rangle} |u|$ or $|f| \sim_{\langle F \rangle} |v|$). Otherwise, the decomposition is said to be **nontrivial**.

Lemma 3.16. Suppose $f \in F(\Lambda)$ is a Θ -element. Then $\langle f \rangle$ is reducible if and only if there exists some generator f' of $\langle f \rangle$ that has a nontrivial Θ -decomposition.

Proof. If $\langle f \rangle$ is reducible, then there exist kernels $\langle u \rangle$ and $\langle v \rangle$ in Θ such that $\langle f \rangle = \langle u \rangle \cap \langle v \rangle$ where $\langle f \rangle \neq \langle u \rangle$ and $\langle f \rangle \neq \langle v \rangle$. Since $\langle u \rangle \cap \langle v \rangle = \langle |u| \wedge |v| \rangle$ we have the nontrivial Θ -decomposition $f' = |u| \wedge |v|$ (which is a generator of $\langle f \rangle$).

Conversely, assume that $f' = |u| \wedge |v|$ is a nontrivial Θ -decomposition for some $f' \sim_{\langle F \rangle} f$. Then $\langle f \rangle = \langle f' \rangle = \langle |u| \wedge |v| \rangle = \langle u \rangle \cap \langle v \rangle$. Since the decomposition $f' = |u| \wedge |v|$ is nontrivial, we have that $u \not\sim_{\langle F \rangle} f'$ and $v \not\sim_{\langle F \rangle} f'$, and thus $\langle u \rangle = \langle u \rangle \neq \langle f' \rangle = \langle f \rangle$. Similarly, $\langle v \rangle \neq \langle f \rangle$. Thus, by definition, $\langle f \rangle$ is reducible. \square

We can equivalently rephrase Lemma 3.16 as follows:

Remark 3.17. f is reducible if and only if some $f' \sim_{\langle F \rangle} f$ has a nontrivial Θ -decomposition.

A question immediately arising from Definition 3.15 and Lemma 3.16 is:

If $f \in F(\Lambda)$ has a nontrivial Θ -decomposition and $g \sim_{\langle F \rangle} f$, does g also have a nontrivial Θ -decomposition? If so, how is this pair of decompositions related?

In the next few paragraphs we provide an answer to both of these questions, for $\Theta = \mathcal{P}(\langle \mathcal{R} \rangle)$.

Remark 3.18. $\sum_{i=1}^k s_i (a_i \wedge b_i)^{d(i)} = \left(\sum_{i=1}^k s_i a_i^{d(i)} \right) \wedge \left(\sum_{i=1}^k s_i b_i^{d(i)} \right)$, $\forall s_1, \dots, s_k, a_1, \dots, a_k, b_1, \dots, b_k \in F(\Lambda)$, and $d(i) \in \mathbb{N}_{\geq 0}$.

Remark 3.19. If $h_1, \dots, h_k \in F(\Lambda)$ such that each $h_i \geq_\nu 1$, then $\sum_{i=1}^k s_i h_i \geq_\nu 1$ for every $s_1, \dots, s_k \in F(\Lambda)$ such that $\sum_{i=1}^k s_i \cong_\nu 1$.

Theorem 3.20. (For $\Theta = \mathcal{P}(\langle \mathcal{R} \rangle)$.) If $\langle f \rangle$ is a (principal) reducible kernel, then there exist Θ -elements $g, h \in F(\Lambda)$ such that $|f| = |g| \wedge |h|$ and $|f| \not\sim_{\langle F \rangle} |g|, |h|$.

Proof. If $\langle f \rangle$ is a principal reducible kernel, then there exists $f' \sim_{\langle F \rangle} f$ such that $f' = |u| \wedge |v| = \min(|u|, |v|)$ for Θ -elements $u, v \in \langle \mathcal{R} \rangle$ with $f' \not\sim_{\langle F \rangle} |u|, |v|$. Then $|f| \in \langle f' \rangle$ since f' is a generator of $\langle f \rangle$, so there exist $s_1, \dots, s_k \in F(\Lambda)$ such that $\sum_{i=1}^k s_i = 1$ and $|f| = \sum_{i=1}^k s_i (f')^{d(i)}$ with $d(i) \in \mathbb{N}_{\geq 0}$. ($d(i) \geq 0$ since $|f| \geq_\nu 1$.) Thus

$$f = \sum_{i=1}^k s_i (|u| \wedge |v|)^{d(i)} = \sum_{i=1}^k s_i (\min(|u|, |v|))^{d(i)} = \min \left(\sum_{i=1}^k s_i |u|^{d(i)}, \sum_{i=1}^k s_i |v|^{d(i)} \right) = |g| \wedge |h|$$

where $g = |g| = \sum_{i=1}^k s_i |u|^{d(i)}$, $h = |h| = \sum_{i=1}^k s_i |v|^{d(i)}$.

Now $\langle |f| \rangle \subseteq \langle |g| \rangle \subseteq \langle |u| \rangle$ and $\langle |f| \rangle \subseteq \langle |h| \rangle \subseteq \langle |v| \rangle$, implying $1_{\text{loc}}(f) \supseteq 1_{\text{loc}}(g) \supseteq 1_{\text{loc}}(u)$ and $1_{\text{loc}}(f) \supseteq 1_{\text{loc}}(h) \supseteq 1_{\text{loc}}(v)$.

We claim that $|g|$ and $|h|$ generate $\langle |u| \rangle$ and $\langle |v| \rangle$, respectively. Indeed, $1_{\text{loc}}(f') = 1_{\text{loc}}(|f|)$, since $f' \sim_{\langle F \rangle} |f|$ and thus for any $\mathbf{a} \in F^{(n)}$, $f'(\mathbf{a}) = 1 \Leftrightarrow |f|(\mathbf{a}) = 1$. Let $s_j(f')^{d(j)}$ be a dominant term of $|f|$ at \mathbf{a} , i.e.,

$$|f| \cong_{\nu} \sum_{i=1}^k s_i(\mathbf{a})(f'(\mathbf{a}))^{d(i)} \cong_{\nu} s_j(\mathbf{a})(f'(\mathbf{a}))^{d(j)}.$$

Then $f(\mathbf{a}) \cong_{\nu} 1 \Leftrightarrow s_j(\mathbf{a})(f'(\mathbf{a}))^{d(j)} \cong_{\nu} 1$. If $f'(\mathbf{a}) \cong_{\nu} 1$, then $(f'(\mathbf{a}))^{d(j)} \cong_{\nu} 1$, so $s_j(\mathbf{a}) \cong_{\nu} 1$. Now, for $\mathbf{a} \in 1_{\text{loc}}(g)$. Then we have

$$g(\mathbf{a}) \cong_{\nu} \sum_{i=1}^k s_i |u|^{d(i)} \cong_{\nu} 1.$$

Let $s_t |u|^{d(t)}$ be a dominant term of g at \mathbf{a} . If $s_t(\mathbf{a}) \cong_{\nu} 1$ then $|u|^{d(t)} \cong_{\nu} 1$ and thus $u = 1$, and $\mathbf{a} \in 1_{\text{loc}}(u)$. Otherwise $s_t(\mathbf{a}) <_{\nu} 1$ (since $\sum_{i=1}^k s_i \cong_{\nu} 1$) and so, by the above, $s_t(f')^{d(t)}$ is not a dominant term of $|f|$ at \mathbf{a} . Thus, for every index j of a dominant term of $|f|$ at \mathbf{a} , we have $j \neq t$ and

$$|u(\mathbf{a})|^{d(j)} \cong_{\nu} s_j(\mathbf{a}) |u(\mathbf{a})|^{d(j)} <_{\nu} s_t(\mathbf{a}) |u(\mathbf{a})|^{d(t)} \cong_{\nu} g(\mathbf{a}) \cong_{\nu} 1.$$

$$(3.3) \quad s_j(\mathbf{a})(f'(\mathbf{a}))^{d(j)} \cong_{\nu} s_j(\mathbf{a})(|u|(\mathbf{a}) \wedge |v|(\mathbf{a}))^{d(j)} \leq s_j(\mathbf{a}) |u(\mathbf{a})|^{d(j)} <_{\nu} 1.$$

On the other hand, $f'(\mathbf{a}) \cong_{\nu} 1$ since $1_{\text{loc}}(f) \supseteq 1_{\text{loc}}(g)$, implying $s_j(\mathbf{a})(f'(\mathbf{a}))^{d(j)} \cong_{\nu} 1$, contradicting (3.3). Hence, $1_{\text{loc}}(g) \subseteq 1_{\text{loc}}(u)$, yielding $1_{\text{loc}}(g) = 1_{\text{loc}}(u)$, which implies that g is a generator of $\langle |u| \rangle = \langle u \rangle$. The proofs for h and $|v|$ are analogous.

Consequently, $g \sim_{\langle F \rangle} |g| \sim_{\langle F \rangle} |u|$ and $h \sim_{\langle F \rangle} |h| \sim_{\langle F \rangle} |v|$. Since $|f| \sim_{\langle F \rangle} f' \not\sim_{\langle F \rangle} |u|, |v|$ we conclude that $|f| \not\sim_{\langle F \rangle} |g|, |h|$. \square

Corollary 3.21. *For $f \in \langle F \rangle$, if $|f| = \bigwedge_{i=1}^s |f_i|$ for $f_i \in \langle F \rangle$, then for any $g \sim_{\langle F \rangle} f$, we have $|g| = \bigwedge_{i=1}^s |g_i|$, with $g_i \sim_{\langle F \rangle} f_i$ for $i = 1, \dots, s$.*

Proof. Iterate Theorem 3.20. \square

Corollary 3.22. *If $\langle f \rangle$ is a kernel in Θ , then $\langle f \rangle$ has a nontrivial decomposition $\langle f \rangle = \langle g \rangle \cap \langle h \rangle$ if and only if $|f|$ has a nontrivial decomposition $|f| = |g'| \wedge |h'|$ with $|g'| \sim_{\langle F \rangle} g$ and $|h'| \sim_{\langle F \rangle} h$.*

Proof. If $|f| = |g'| \wedge |h'|$ then, since $|g'| \sim_{\langle F \rangle} g$ and $|h'| \sim_{\langle F \rangle} h$ we have

$$\langle f \rangle = \langle |f| \rangle = \langle |g'| \wedge |h'| \rangle = \langle |g'| \rangle \cap \langle |h'| \rangle = \langle g \rangle \cap \langle h \rangle.$$

The converse is seen as in the proof of Theorem 3.20. \square

Corollary 3.22 provides a Θ -decomposition of $|f|$, for every generator f of a reducible kernel in Θ .

Remark 3.23. By [23, Corollary 4.1.25],

$$\langle f \rangle \cap \langle g \rangle = \langle (f + f^*) \wedge (g + g^*) \rangle = \langle |f| \wedge |g| \rangle.$$

But, in fact, $\langle f \rangle \cap \langle g \rangle = \langle f' \rangle \cap \langle g' \rangle$ for any $g' \sim_{\langle F \rangle} g$ and $h' \sim_{\langle F \rangle} h$, so we could take $|g'| \wedge |f'|$ instead of $|g| \wedge |f|$ on the righthand side of the equality, e.g., $\langle |f|^k \wedge |g|^m \rangle$ for any $m, k \in \mathbb{Z} \setminus \{0\}$.

Definition 3.24. Let \mathbb{S} be a semifield and let $a, b \in \mathbb{S}$. We say that a and b are $\langle F \rangle$ -**comparable** if there exist some $a' \sim_{\langle F \rangle} a$ and $b' \sim_{\langle F \rangle} b$ such that $|a'| \leq |b'|$ or $|b'| \leq |a'|$.

Since $|g| \wedge |h| = \min(|g|, |h|)$ we can utilize Remark 3.23 to get the following observation:

Proposition 3.25. *A Θ -decomposition $f \sim_{\langle F \rangle} |g| \wedge |h| \in F(\Lambda)$ is nontrivial if and only if the Θ -elements g and h are not $\langle F \rangle$ -comparable.*

Proof. If g and h are $\langle F \rangle$ -comparable, then there exist some $g' \sim_{\langle F \rangle} g$ and $h' \sim_{\langle F \rangle} h$ such that $|g'| \geq_\nu |h'|$ or $|h'| \geq_\nu |g'|$. Without loss of generality, assume that $|g'| \geq_\nu |h'|$. Then $\langle |g| \wedge |h| \rangle = \langle |g| \rangle \cap \langle |h| \rangle = \langle |g'| \rangle \cap \langle |h'| \rangle = \langle |g'| \wedge |h'| \rangle = \langle \min(|g'|, |h'|) \rangle = \langle |g'| \rangle = \langle g' \rangle = \langle g \rangle$. Thus $\langle f \rangle = \langle g \rangle$ so $f \sim_{\langle F \rangle} g$ yielding that the decomposition is trivial.

Conversely, if g and h are not $\langle F \rangle$ -comparable then we claim that $f \not\sim_{\langle F \rangle} g$ and $f \not\sim_{\langle F \rangle} h$. We must show that $\langle h \rangle \not\subseteq \langle g \rangle$ and $\langle g \rangle \not\subseteq \langle h \rangle$ respectively. So assume that $\langle h \rangle \supseteq \langle g \rangle$. In view of Lemma 2.48 and [23, Proposition 4.1.13], there exists some $f' \sim f$ such that $f' = |h|^k \wedge g$. Note that $|h|^k \geq 1$ and $g \geq 1$ so $f' = |h|^k \wedge g \geq 1$ and thus $|f'| = f'$. Finally,

$$|f'| = |h|^k \wedge g \Leftrightarrow |f'| \leq |h|^k \Leftrightarrow |f'| \in \langle |h| \rangle \Leftrightarrow f' \in \langle h \rangle \Leftrightarrow f \in \langle h \rangle.$$

□

3.3. Convex dependence.

Definition 3.26. An HS-fraction f of $F(\Lambda)$ is **F -convexly dependent** on a set A of HS-fractions if

$$(3.4) \quad f \in \langle \{g : g \in A\} \rangle \cdot \langle F \rangle;$$

otherwise f is said to be **F -convexly-independent** of A . The set A is said to be **F -convexly independent** if f is F -convexly independent of $A \setminus \{f\}$, for every $f \in A$. If $\{a_1, \dots, a_n\}$ is F -convexly dependent, then we also say that a_1, \dots, a_n are F -convexly dependent.

Note that under the assumption that $g \in \langle F \rangle \setminus \{1\}$ for some $g \in A$, the condition in (3.4) simplifies to $f \in \langle \{g : g \in A\} \rangle$.

Remark 3.27. By definition, an HS-fraction f is F -convexly dependent on HS-fractions $\{g_1, \dots, g_t\}$ if and only if

$$\langle |f| \rangle = \langle f \rangle \subseteq \langle |g_1|, \dots, |g_t| \rangle \cdot \langle F \rangle = \left\langle \sum_{i=1}^t |g_i| \right\rangle \cdot \langle F \rangle = \left\langle \sum_{i=1}^t |g_i| + |\alpha| \right\rangle,$$

for any element α of F for which $\alpha^\nu \neq 1^\nu$.

Example 3.28. For any $\alpha \in F$ and any $f \in F(\Lambda)$,

$$|\alpha f| \leq |f|^2 + |\alpha|^2 = (|f| + |\alpha|)^2.$$

Thus $\alpha f \in \langle (|f| + |\alpha|)^2 \rangle = \langle |f| + |\alpha| \rangle = \langle f \rangle \cdot \langle F \rangle$. In particular, if f is an HS-fraction, then αf is F -convexly dependent on f .

As a consequence of Lemma 2.48, if two \mathcal{L} -monomials f, g , satisfy $g \in \langle f \rangle$, then $\langle g \rangle = \langle f \rangle$. In other words, either $\langle g \rangle = \langle f \rangle$ or $\langle g \rangle \not\subseteq \langle f \rangle$ and $\langle f \rangle \not\subseteq \langle g \rangle$. This motivates us to restrict the convex dependence relation to the set of \mathcal{L} -monomials. This will be justified later by showing that for each F -convexly independent subset of HS-fractions of order t in $F(\Lambda)$, there exists an F -convexly independent subset of \mathcal{L} -monomials having order $\geq t$ in $F(\Lambda)$. Let us see that convex-dependent is an abstract dependence relation.

Proposition 3.29. Let $A, A_1 \subset F(\Lambda)$ be sets of HS-fractions, and let f be an HS-fraction.

- (1) If $f \in A$, then f is F -convexly-dependent on A .
- (2) If f is F -convexly dependent on A and each $a \in A$ is F -convexly dependent on A_1 , then f is F -convexly dependent on A_1 .
- (3) If f is F -convexly dependent on A , then f is F -convexly dependent on A_0 for some finite subset A_0 of A .

Proof. (1) $f \in \langle A \rangle \subseteq \langle A \rangle \cdot \langle F \rangle$.

(2) $\langle A \rangle \subseteq \langle A_1 \rangle \cdot \langle F \rangle$ since a is convexly-dependent on A_1 for each $a \in A$. If f is F -convexly dependent on A , then $f \in \langle A \rangle \cdot \langle F \rangle \subseteq \langle A_1 \rangle \cdot \langle F \rangle$, so, f is F -convexly dependent on A_1 .

(3) $a \in \langle A \rangle \cdot \langle F \rangle$, so by Proposition 2.11 there exist some $s_1, \dots, s_k \in F(\Lambda)$ and $g_1, \dots, g_k \in G(A \cup F) \subset \langle A \rangle \cdot \langle F \rangle$, where $G(A \cup F)$ is the group generated by $A \cup F$, such that $\sum_{i=1}^k s_i = 1$ and $a = \sum_{i=1}^k s_i g_i^{d(i)}$ with $d(i) \in \mathbb{Z}$. Thus $a \in \langle g_1, \dots, g_k \rangle$ and $A_0 = \{g_1, \dots, g_k\}$. □

From now on, we assume that the ν -semifield[†] F is divisible.

Proposition 3.30 (Steinitz exchange axiom). Let $S = \{b_1, \dots, b_t\} \subset \text{HP}(F(\Lambda))$ and let f and b be elements of $\text{HP}(F(\Lambda))$. If f is F -convexly-dependent on $S \cup \{b\}$ and f is F -convexly independent of S , then b is F -convexly-dependent on $S \cup \{f\}$.

Proof. We may assume that $\alpha \in S$ for some $\alpha \in F$. Since f is F -convexly independent of S , by definition $f \notin \langle S \rangle$ this implies that $\langle S \rangle \subset \langle S \rangle \cdot \langle f \rangle$ (for otherwise $\langle f \rangle \subseteq \langle S \rangle$ yielding that f is F -convexly dependent on S). Since f is F -convexly-dependent on $S \cup \{b\}$, we have that $f \in \langle S \cup \{b\} \rangle = \langle S \rangle \cdot \langle b \rangle$. In particular, we get that $b \notin \langle S \rangle \cdot \langle F \rangle$ for otherwise f would be dependent on S . Consider the quotient map $\phi : F(\Lambda) \rightarrow F(\Lambda)/\langle S \rangle$. Since ϕ is a semifield epimorphism and $f, b \notin \langle S \rangle \cdot \langle F \rangle = \phi^{-1}(\langle F \rangle)$, we have that $\phi(f)$ and $\phi(b)$ are not in F thus are \mathcal{L} -monomials in the semifield $\text{Im}(\phi) = F(\Lambda)/\langle S \rangle$. By the above, $\phi(f) \neq 1$ and $\phi(f) \in \phi(\langle b \rangle) = \langle \phi(b) \rangle$. Thus, $\langle \phi(f) \rangle = \langle \phi(b) \rangle$ by Lemma 2.48. So $\langle S \rangle \cdot \langle f \rangle = \phi^{-1}(\langle \phi(f) \rangle) = \phi^{-1}(\langle \phi(b) \rangle) = \langle S \rangle \cdot \langle b \rangle$, consequently $b \in \langle S \rangle \cdot \langle b \rangle = \langle S \rangle \cdot \langle f \rangle = \langle S \cup \{f\} \rangle$, i.e., b is F -convexly-dependent on $S \cup \{f\}$. \square

Definition 3.31. Let $A \subseteq \text{HP}(F(\Lambda))$. The **convex span** of A over F is the set

$$(3.5) \quad \text{Conv}_F(A) = \{a \in \text{HP}(F(\Lambda)) : a \text{ is } F\text{-convexly dependent on } A\}.$$

For a semifield[†] $K \subseteq F(\Lambda)$ such that $F \subseteq K$, a set $A \subseteq \text{HP}(F(\Lambda))$ is said to **convexly span** K over F if

$$\text{HP}(K) = \text{Conv}_F(A).$$

Remark 3.32. $\text{Conv}(\{f_1, \dots, f_m\}) = \langle f_1, \dots, f_m \rangle \cdot \langle F \rangle$.

In view of Propositions 3.29 and 3.30, convex dependence on $\text{HP}(F(\Lambda))$ is an abstract dependence relation. Then by [25, Chapter 6], we have:

Corollary 3.33. *Let $V \subset \text{HP}(F(\Lambda))$. Then V contains a basis $B_V \subset V$, which is a maximal convexly independent subset of unique cardinality such that*

$$\text{Conv}(B_V) = \text{Conv}(V).$$

Example 3.34. By Lemma 2.32, the maximal kernels in $\mathcal{P}(F(\Lambda))$ are HS-fractions of the form $L_{(\alpha_1, \dots, \alpha_n)} = \langle \alpha_1 x_1, \dots, \alpha_n x_n \rangle$ for any $\alpha_1, \dots, \alpha_n \in F$. In view of Corollary 2.31,

$$F(\Lambda) = \text{Conv}(\{\alpha_1 x_1, \dots, \alpha_n x_n\}),$$

i.e., $\{\alpha_1 x_1, \dots, \alpha_n x_n\}$ convexly spans $F(\Lambda)$ over F . Now $\alpha_k x_k \notin \langle \bigcup_{j \neq k} \alpha_j x_j \rangle \cdot \langle F \rangle$, since there are no order relations between $\alpha_i x_i$ and the elements of $\{\alpha_j x_j : j \neq i\} \cup \{\alpha : \alpha \in F\}$. Thus, for arbitrary $\alpha_1, \dots, \alpha_n \in F$, $\{\alpha_1 x_1, \dots, \alpha_n x_n\}$ is F -convexly independent, constituting a basis for $F(\Lambda)$.

Definition 3.35. Let $V \subset \text{HP}(F(\Lambda))$ be a set of \mathcal{L} -monomials. We define the **convex dimension** of V , $d_{\text{conv}}(V)$, to be $|B|$ where B is a basis for V .

Example 3.36. $d_{\text{conv}}(F(\Lambda)) = n$, by Example 3.34.

Remark 3.37. If $S \subset \text{HP}(F(\Lambda))$, then for any $f, g \in F(\Lambda)$ such that $f, g \in \text{Conv}(S)$

$$|f| + |g| \in \text{Conv}(S) \quad \text{and} \quad |f| \wedge |g| \in \text{Conv}(S).$$

Proof. First we prove that $|f| + |g| \in \text{Conv}(S)$. Since $\langle f \rangle \subseteq \langle S \rangle \cdot \langle F \rangle$ and $\langle g \rangle \subseteq \langle S \rangle \cdot \langle F \rangle$, we have $\langle f, g \rangle = \langle |f| + |g| \rangle = \langle f \rangle \cdot \langle g \rangle \subseteq \langle S \rangle \cdot \langle F \rangle$. $|f| \wedge |g| \in \text{Conv}(S)$, since $\langle |f| \wedge |g| \rangle = \langle f \rangle \cap \langle g \rangle \subseteq \langle g \rangle \subseteq \langle S \rangle \cdot \langle F \rangle$. \square

Remark 3.38. If K is an HS-kernel, then K is generated by an HS-fraction $f \in F(\Lambda)$ of the form $f = \sum_{i=1}^t |f_i|$ where f_1, \dots, f_t are \mathcal{L} -monomials. So,

$$\begin{aligned} \text{Conv}(K) &= \langle F \rangle \cdot K = \langle F \rangle \cdot \langle f \rangle = \langle F \rangle \cdot \left\langle \sum_{i=1}^t |f_i| \right\rangle = \langle F \rangle \cdot \prod_{i=1}^t \langle f_i \rangle \\ &= \langle F \rangle \cdot \langle f_1, \dots, f_t \rangle \end{aligned}$$

and so, $\{f_1, \dots, f_t\}$ convexly spans $\langle F \rangle \cdot K$.

Remark 3.39. Let f be an HS-fraction. Then $f \sim_{\langle F \rangle} \sum_{i=1}^t |f_i|$ where f_i are \mathcal{L} -monomials. Hence f is F -convexly dependent on $\{f_1, \dots, f_t\}$, since $\langle f \rangle = \prod_{i=1}^t \langle f_i \rangle = \langle \{f_1, \dots, f_t\} \rangle$.

Lemma 3.40. *Suppose $\{b_1, \dots, b_m\}$ is a set of HS-fractions, such that $b_i \sim_{\langle F \rangle} \sum_{j=1}^{t_i} |f_{i,j}|$, where $f_{i,j}$ are \mathcal{L} -monomials. Then b_1 is F -convexly dependent on $\{b_2, \dots, b_m\}$ if and only if all of its summands $f_{1,r}$ for $1 \leq r \leq t_1$ are F -convexly dependent on $\{b_2, \dots, b_m\}$.*

Proof. If b_1 is F -convexly dependent on $\{b_2, \dots, b_m\}$, then

$$\prod_{j=1}^{t_1} \langle f_{1,j} \rangle = \left\langle \sum_{j=1}^{t_1} |f_{1,j}| \right\rangle = \langle b_1 \rangle \subseteq \langle \{b_1, \dots, b_m\} \rangle.$$

Hence $f_{1,r} \in \prod_{j=1}^{t_1} \langle f_{1,j} \rangle$ is F -convexly dependent on $\{b_1, \dots, b_m\}$ and by Remark 3.39 $f_{1,r}$ is F -convexly dependent on $\{f_{i,j} : 2 \leq i \leq m; 1 \leq j \leq t_i\}$. Conversely, if each $f_{1,r}$ is F -convexly dependent on $\{b_2, \dots, b_m\}$ for $1 \leq r \leq t_1$, then there exist some k_1, \dots, k_{t_1} such that $|f_{1,r}| \leq \sum_{i=2}^m \sum_{j=1}^{t_i} |f_{i,j}|^{k_r}$. Hence b_1 is F -convexly dependent on $\{b_2, \dots, b_m\}$, by Corollary 2.18. \square

Lemma 3.41. *Let $V = \{f_1, \dots, f_m\}$ be a F -convexly independent set of HS-fractions, with $f_i \sim_{\langle F \rangle} \sum_{j=1}^{t_i} |f_{i,j}|$ for \mathcal{L} -monomials $f_{i,j}$. Then there exists an F -convexly independent subset*

$$S_0 \subseteq S = \{f_{i,j} : 1 \leq i \leq m; 1 \leq j \leq t_i\}$$

such that $|S_0| \geq |V|$ and $\text{Conv}(S_0) = \text{Conv}(V)$.

Proof. By Remark 3.39, f_i is dependent on the set of \mathcal{L} -monomials $\{f_{i,j} : 1 \leq j \leq t_i\} \subset S$ for each $1 \leq i \leq m$, implying $\text{Conv}(S) = \text{Conv}(V)$. By Corollary 3.33, S contains a maximal F -convexly independent subset S_0 such that $\text{Conv}(S_0) = \text{Conv}(S)$ which, by Lemma 3.40, we can shrink down to a base. \square

Lemma 3.42. *The following hold for an \mathcal{L} -monomial f :*

- (1) $\langle F \rangle \not\subseteq \langle f \rangle$.
- (2) *If $F(\Lambda)$ is not bounded, then $\langle f \rangle \not\subseteq \langle F \rangle$.*

Proof. By definition, an \mathcal{L} -monomial is not bounded from below. Thus $\langle f \rangle \cap F = \{1\}$, yielding $\langle F \rangle \not\subseteq \langle f \rangle$. For the second assertion, an HP-kernel is not bounded when $F(\Lambda)$ is not bounded, so $\langle f \rangle \not\subseteq \langle F \rangle$. \square

A direct consequence of Lemma 3.42 is:

Lemma 3.43. *If $F(\Lambda)$ is not bounded, then any nontrivial HS-kernel (i.e., $\neq \langle 1 \rangle$) is F -convexly independent.*

Proof. By Lemma 3.42 the assertion is true for HP-kernels, and thus for HS-kernels, since every HS-kernel contains some HP-kernel. \square

3.4. Computing convex dimension.

Having justified our restriction to \mathcal{L} -monomials, we move ahead with computing lengths of chains.

Remark 3.44. Let K be an HS-kernel of $F(\Lambda)$. By definition there are \mathcal{L} -monomials $f_1, \dots, f_t \in \text{HP}(K)$ such that $K = \langle \sum_{i=1}^t |f_i| \rangle$. By Remark 3.38, $\text{Conv}(K)$ is convexly spanned by f_1, \dots, f_t . Now, since $\text{Conv}(K) = \text{Conv}(f_1, \dots, f_t)$ and $\{f_1, \dots, f_t\} \subset \text{HP}(K) \subset \text{HP}(F(\Lambda))$, by Corollary 3.33, $\{f_1, \dots, f_t\}$ contains a basis $B = \{b_1, \dots, b_s\} \subset \{f_1, \dots, f_t\}$ of F -convexly independent elements, where $s = d_{\text{conv}}(K)$, such that $\text{Conv}(B) = \text{Conv}(f_1, \dots, f_t) = \text{Conv}(K)$.

Proposition 3.45. *For any order kernel o of $F(\Lambda)$, if \mathcal{L} -monomials h_1, \dots, h_t are F -convexly dependent, then the images of h_1, \dots, h_t is F -convexly dependent (in the quotient semifield[†] $F(\Lambda)/o$).*

Proof. Denote by $\phi_o : F(\Lambda) \rightarrow F(\Lambda)/o$ the quotient F -homomorphism. Then $\phi_o(\langle F \rangle) = \langle \phi_o(F) \rangle = \langle F \rangle_{F(\Lambda)/o}$. Now, if h_1, \dots, h_t are F -convexly dependent then there exist some j , say without loss of generality $j = 1$, such that $h_1 \in \langle h_2, \dots, h_t \rangle \cdot \langle F \rangle$. By assumption and Proposition 2.17,

$$\begin{aligned} \phi_o(h_1) &\in \phi_o(\langle h_2, \dots, h_t, \alpha \rangle) \\ &= \langle \phi_o(h_2), \dots, \phi_o(h_t), \phi_o(\alpha) \rangle = \langle \phi_o(h_2), \dots, \phi_o(h_t), \alpha \rangle \\ &= \langle \phi_o(h_2), \dots, \phi_o(h_t) \rangle \cdot \langle F \rangle \end{aligned}$$

Thus $\phi_o(h_1)$ is F -convexly dependent on $\{\phi_o(h_2), \dots, \phi_o(h_t)\}$. \square

Conversely, we have:

Lemma 3.46. *For any order kernel o of $F(\Lambda)$, and any set $\{h_1, \dots, h_t\}$ of \mathcal{L} -monomials, if $\phi_o(h_1), \dots, \phi_o(h_t)$ are F -convexly dependent in the quotient semifield[†] $F(\Lambda)/o$ and $\sum_{i=1}^t \phi_o(|h_i|) \cap F = \{1\}$, then h_1, \dots, h_t are F -convexly dependent in $F(\Lambda)$.*

Proof. Note that $\sum_{i=1}^t \phi_o(|h_i|) \cap F = \{1\}$ if and only if $\bigcap_{i=1}^t 1\text{-set}(h_i) \cap 1\text{-set}(o) \neq \emptyset$. Translating the variables by a point $a \in \bigcap_{i=1}^t 1\text{-set}(h_i) \cap 1\text{-set}(o)$, we may assume that the constant coefficient of each \mathcal{L} -monomial h_i is 1. Assume that $\phi_o(h_1), \dots, \phi_o(h_t)$ are F -convexly dependent. We may assume that $\phi_o(h_1)$ is F -convexly dependent on $\phi_o(h_2), \dots, \phi_o(h_t)$. This means by Definition 3.26 that we can take $h_{t+1} \in F$ for which $\phi_o(h_1) \in \langle \phi_o(h_2), \dots, \phi_o(h_t), \phi_o(h_{t+1}) \rangle$. Taking the pre-images of the quotient map yields

$$\langle h_1 \rangle \cdot o \subseteq \langle h_2, \dots, h_t, h_{t+1} \rangle \cdot o.$$

Take an \mathcal{L} -monomial g such that $1 + g$ generates o . By Corollary 2.18, there exists some $k \in \mathbb{N}$ such that

$$(3.6) \quad |h_1| + |1 + g| \leq_\nu (|h_2| + \dots + |h_{t+1}| + |1 + g|)^k = |h_2|^k + \dots + |h_{t+1}|^k + |1 + g|^k.$$

As $1 + g \geq_\nu 1$ we have that $|1 + g| \cong_\nu 1 + g$, and the right hand side of Equation (3.6) equals

$$|h_2|^k + \dots + |h_{t+1}|^k + (1 + g)^k \cong_\nu |h_2|^k + \dots + |h_{t+1}|^k + 1 + g^k \cong_\nu |h_2|^k + \dots + |h_{t+1}|^k + g^k.$$

The last equality is due to the fact that $\sum |h_i|^k \geq_\nu 1$ so that 1 is absorbed. The same argument, applied to the left hand side of Equation (3.6), yields that

$$(3.7) \quad |h_1| + g \leq_\nu |h_2|^k + \dots + |h_{t+1}|^k + g^k.$$

Assume on the contrary that h_1 is F -convexly independent of $\{h_2, \dots, h_t\}$. Then

$$\langle h_1 \rangle \not\subseteq \langle h_2, \dots, h_{t+1} \rangle \cong_\nu \left\langle \sum_{i=2}^{t+1} |h_i| \right\rangle.$$

Thus for any $m \in \mathbb{N}$ there exists some $\mathbf{a}_m \in F^{(n)}$ such that

$$|h_1(\mathbf{a}_m)| >_\nu \left| \sum_{i=2}^{t+1} |h_i(\mathbf{a}_m)| \right|^m \cong_\nu \sum_{i=2}^{t+1} |h_i(\mathbf{a}_m)|^m.$$

Thus by equation (3.7) and the last observation we get that

$$\sum_{i=2}^t |h_i(\mathbf{a}_m)|^m + g(\mathbf{a}_m) <_\nu |h_1(\mathbf{a}_m)| + g(\mathbf{a}_m) \leq_\nu \sum_{i=2}^t |h_i(\mathbf{a}_m)|^k + g(\mathbf{a}_m)^k,$$

i.e., there exists some fixed $k \in \mathbb{N}$ such that for any $m \in \mathbb{N}$,

$$(3.8) \quad \sum_{i=2}^t |h_i(\mathbf{a}_m)|^m <_\nu \sum_{i=2}^t |h_i(\mathbf{a}_m)|^k + g(\mathbf{a}_m)^k.$$

For $m > k$, since $|\gamma|^k \leq_\nu |\gamma|^m$ for any $\gamma \in F$, we get that $\sum_{i=2}^t |h_i(\mathbf{a}_m)|^m \geq_\nu \sum_{i=2}^t |h_i(\mathbf{a}_m)|^k$. Write

$$g^k = g(1)g'.$$

Since g^k is an HP-kernel, $g(1)$ is the constant coefficient of g and g' is a Laurent monomial with coefficient 1.

According to the way \mathbf{a}_m were chosen, $\sum_{i=2}^t |h_i(\mathbf{a}_m)| > 1$ and $\sum_{i=2}^t |h_i(\mathbf{a}_m)|^m <_\nu g(\mathbf{a}_m)^k$, and thus $g(1) <_\nu g'(\mathbf{a}_{m_0})$ for large enough m_0 . But $g'(\mathbf{a}_{m_0}^{-1}) \cong_\nu g'(\mathbf{a}_m)^{-1}$ so

$$g^k(\mathbf{a}_{m_0}^{-1}) \cong_\nu g(1)g'(\mathbf{a}_{m_0}^{-1}) \cong_\nu g(1)g'(\mathbf{a}_{m_0})^{-1} <_\nu 1.$$

Thus (3.8) yields $\sum_{i=2}^t |h_i(\mathbf{a}_{m_0}^{-1})|^m <_\nu \sum_{i=2}^t |h_i(\mathbf{a}_{m_0}^{-1})|^k$, a contradiction. \square

Proposition 3.47. *Let R be a region kernel of $F(\Lambda)$. Let $\{h_1, \dots, h_t\}$ be a set of \mathcal{L} -monomials such that $(R \cdot \langle h_1, \dots, h_t \rangle) \cap F = \{1\}$. Then $h_1 \cdot R, \dots, h_t \cdot R$ are F -convexly dependent in the quotient semifield[†] $F(\Lambda)/R$ if and only if h_1, \dots, h_t are F -convexly dependent in $F(\Lambda)$.*

Proof. The ‘if’ part of the assertion follows from Proposition 3.45. Since $R = \prod_{i=1}^m o_i$ for suitable order kernels $\{o_i\}_{i=1}^m$, the ‘only if’ part follows from Lemma 3.46 applied repeatedly to each of these o_i ’s. \square

Proposition 3.48. *Let $R \in \mathcal{P}(F(\Lambda))$ be a region kernel. Then, for any set L of HS-fractions,*

$$d_{\text{conv}}(L) = d_{\text{conv}}(L \cdot R),$$

the right side taken in $F(\Lambda)/R$.

Proof. $d_{\text{conv}}(L) \leq d_{\text{conv}}(R \cdot L)$ since $L \subseteq R \cdot L$. For the reverse inequality, let $\phi_R : F(\Lambda) \rightarrow F(\Lambda)/R$ be the quotient map. Since L is a sub-semifield[†] of $F(\Lambda)$, $\phi_R^{-1}(\phi_R(L)) = R \cdot \langle L \rangle$ by Theorem 2.14, and $d_{\text{conv}}(L) \geq d_{\text{conv}}(\phi_R(L))$ by Proposition 3.45, while $d_{\text{conv}}(\phi_R(L)) \geq d_{\text{conv}}(\phi_R^{-1}(\phi_R(L)))$ by Lemma 3.46. Thus $d_{\text{conv}}(L) \geq d_{\text{conv}}(R \cdot L)$. \square

In this way we see that $K \mapsto \Omega(K)$ yields a homomorphism of kernels. Hence Ω is a natural map in the sense of Definition 2.39, and we can apply Theorem 2.40.

Remark 3.49. Let R be a region kernel and let

$$A = \{\langle g \rangle : \langle g \rangle \cdot \langle F \rangle \supseteq R \cdot \langle F \rangle\}.$$

Then $d_{\text{conv}}(F(\Lambda)/R) = d_{\text{conv}}(A)$, in view of Remark 3.12 and Proposition 3.47. As $L_a \in A$ for any $a \in 1\text{-set}(R) \neq \emptyset$ and $d_{\text{conv}}(L_a) = d_{\text{conv}}(F(\Lambda))$, we conclude that $d_{\text{conv}}(A) = d_{\text{conv}}(F(\Lambda))$.

We are ready for catenarity of d_{conv} .

Theorem 3.50. *If R is a region kernel and L is an HS-kernel of $F(\Lambda)$, then*

$$d_{\text{conv}}(F(\Lambda)/LR) = d_{\text{conv}}(F(\Lambda)) - d_{\text{conv}}(L).$$

In particular,

$$d_{\text{conv}}(F(\Lambda)/LR) = n - d_{\text{conv}}(L).$$

Proof. $F(\Lambda)/LR \cong (F(\Lambda)/R)/(L \cdot R/R)$, by the third isomorphism theorem. Choose a basis for $\text{HP}(F(\Lambda)/R)$ containing a basis for $\text{HP}(L \cdot R/R)$. Then $d_{\text{conv}}(L \cdot R/R) = d_{\text{conv}}(\phi_R(L))$, by Remark 3.49. But $L \cdot R \cap F = \{1\}$. Hence, by Proposition 3.47, $d_{\text{conv}}(\phi_R(L)) = d_{\text{conv}}(L)$. So

$$d_{\text{conv}}(F(\Lambda)/LR) = d_{\text{conv}}(A) - d_{\text{conv}}(L) = d_{\text{conv}}(F(\Lambda)) - d_{\text{conv}}(L).$$

Thus,

$$d_{\text{conv}}(F(\Lambda)/LR) = d_{\text{conv}}(A) - d_{\text{conv}}(L) = n - d_{\text{conv}}(L).$$

□

Proposition 3.51. *Let L be an HS-kernel in $F(\Lambda)$ with $1\text{-set}(L) \neq \emptyset$. Let $\{h_1, \dots, h_t\}$ be a set of \mathcal{L} -monomials in $\text{HSpec}(F(\Lambda))$ such that $\text{Conv}(h_1, \dots, h_t) = \text{Conv}(L)$ and let $L_i = \langle h_i \rangle$. Then the chain*

$$(3.9) \quad L = \prod_{i=1}^u L_i \supseteq \prod_{i=1}^{u-1} L_i \supseteq \dots \supseteq L_1 \supseteq \langle 1 \rangle.$$

of HS-kernels is strictly descending if and only if h_1, \dots, h_u are F -convexly independent.

Proof. (\Rightarrow) If h_u is F -convexly dependent on $\{h_1, \dots, h_{u-1}\}$, then $L_u = \langle h_u \rangle \subseteq \prod_{i=1}^{u-1} L_i \cdot \langle F \rangle$. Assume that $L_u = \langle h_u \rangle \not\subseteq \prod_{i=1}^{u-1} L_i$. Then $\langle F \rangle \subseteq \prod_{i=1}^u L_i$, implying that $\prod_{i=1}^u L_i$ is not an HS-kernel. Thus $L_u = \langle h_u \rangle \subseteq \prod_{i=1}^{u-1} L_i$, and the chain is not strictly descending.

(\Leftarrow) $1\text{-set}(\prod_{i=1}^t L_i) \subseteq 1\text{-set}(L) \neq \emptyset$ for every $0 \leq t \leq u$, implying that $(\prod_{i=1}^t L_i) \cap F = \{1\}$ for every $0 \leq t \leq u$ (for otherwise $1\text{-set}(\prod_{i=1}^t L_i) = \emptyset$). If $\{h_1, \dots, h_u\}$ is F -convexly independent then $L_u = \langle h_u \rangle \not\subseteq \prod_{i=1}^{u-1} L_i \cdot \langle F \rangle$. By induction, the chain (3.9) is strictly descending.

□

Theorem 3.52. *If $L \in \text{HSpec}(F(\Lambda))$, then $\text{hgt}(L) = d_{\text{conv}}(L)$, cf. Definition 3.11. Moreover, every factor of a descending chain of HS-kernels of maximal length is an HP-kernel.*

Proof. By Proposition 3.51, the maximal length of a chain of HS-kernels descending from an HS-kernel L equals the number of elements in a basis of $\text{Conv}(L)$; thus the chain is of unique length $d_{\text{conv}}(L)$, i.e., $\text{hgt}(L) = d_{\text{conv}}(L)$. Moreover, by Theorem 2.14(2),

$$\prod_{i=1}^j L_i / \prod_{i=1}^{j-1} L_i \cong L_j / \left(L_j \cap \prod_{i=1}^{j-1} L_i \right).$$

Furthermore

$$\left(L_j \cdot \left(L_j \cap \left(\prod_{i=1}^{j-1} L_i \right) \right) \right) \cap F = \{1\},$$

since $L_j \cdot (L_j \cap \prod_{i=1}^{j-1} L_i) = L_j \cap \prod_{i=1}^j L_i \subset \prod_{i=1}^j L_i$ and $(\prod_{i=1}^j L_i) \cap F = \{1\}$. So the image of the HP-kernel L_j in $F(\Lambda)/(L_j \cap (\prod_{i=1}^{j-1} L_i))$ is an HP-kernel. Thus, every factor of the chain is an HP-kernel. □

Corollary 3.53. $\text{Hdim}(F(\Lambda)) = d_{\text{conv}}(F(\Lambda)) = n$.

Remark 3.54. If $R_1 \cap R_2 \cap \cdots \cap R_t = \{1\}$, then $F(\Lambda)$ is a subdirect product

$$F(\Lambda) = F(\Lambda)/(R_1 \cap R_2 \cap \cdots \cap R_t) \hookrightarrow \prod_{i=1}^t F(\Lambda)/R_i.$$

Then for any kernel K of $F(\Lambda)$, $R_1 \cap R_2 \cap \cdots \cap R_t \cap K = \bigcap_{i=1}^t (R_i \cap K) = \{1\}$ and, since K itself is an idempotent semifield[†],

$$K = K / \bigcap_{i=1}^t (R_i \cap K) \cong \prod_{i=1}^t K / (R_i \cap K) \cong \prod_{i=1}^t R_i K / R_i.$$

3.5. Summary.

In conclusion, for every principal regular kernel $\langle f \rangle \in P(F(\Lambda))$, we have obtained explicit region kernels $\{R_{1,1}, \dots, R_{1,s}, R_{2,1}, \dots, R_{2,t}\}$ having trivial intersection, such that

$$\langle f \rangle = \bigcap_{i=1}^s K_i \cap \bigcap_{j=1}^t N_j$$

where $K_i = L_i \cdot R_{1,i}$ for $i = 1, \dots, s$ and appropriate HS-kernels L_i and $N_j = B_j \cdot R_{2,j}$ for $j = 1, \dots, t$ and appropriate bounded from below kernels B_j . If $\langle f \rangle \in \mathcal{P}(\langle F \rangle)$, then, in view of Theorem 2.55 we can take $B_j = \langle F \rangle$ for every $j = 1, \dots, t$. Note that over the various regions in $F^{(n)}$ corresponding to the region kernels $R_{i,j}$, f is locally represented by distinct HS-fractions in $\text{HSpec}(F(\Lambda))$. In fact each region is defined so that the local HS-representation of f is given over the entire region. Thus the $R_{i,j}$'s defining the partition of the space can be obtained as a minimal set of regions over each of which $\langle f \rangle$ takes the form of an HS-kernel.

For each $j = 1, \dots, t$, $d_{\text{conv}}(N_j) = \text{Hdim}(N_j) = 0$, since N_j contains no elements of $\text{HP}(F(\Lambda))$, implying

$$d_{\text{conv}}(F(\Lambda)/N_j) = \text{Hdim}(F(\Lambda)/N_j) = n.$$

For each $i = 1, \dots, s$, $d_{\text{conv}}(K_i) = d_{\text{conv}}(L_i) = \text{Hdim}(L_i) \geq 1$, implying

$$d_{\text{conv}}(F(\Lambda)/K_i) = \text{Hdim}(F(\Lambda)/K_i) = n - \text{Hdim}(L_i) < n.$$

Remark 3.55. In view of the discussion in [23, §9.2], each term $F(\Lambda)/L_i$ corresponds to the linear subspace of $F^{(n)}$ (in logarithmic scale) defined by the linear constraints endowed on the quotient $F(\Lambda)/L_i$ by the HS-kernel L_i . One can think of these terms as an algebraic description of the affine subspaces locally comprising $\text{1-set}(f)$.

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